

NONCROSSED PRODUCT MATRIX SUBRINGS AND IDEALS OF GRADED RINGS

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ABSTRACT. We show that if a groupoid graded ring has a certain nonzero ideal property and the principal component of the ring is commutative, then the intersection of a nonzero twosided ideal of the ring with the commutant of the principal component of the ring is nonzero. Furthermore, we show that for a skew groupoid ring with commutative principal component, the principal component is maximal commutative if and only if it is intersected nontrivially by each nonzero ideal of the skew groupoid ring. We also determine the center of strongly groupoid graded rings in terms of an action on the ring induced by the grading. In the end of the article, we show that, given a finite groupoid G , which has a nonidentity morphism, there is a ring, strongly graded by G , which is not a crossed product over G .

1. INTRODUCTION

Let R be a ring. By this we always mean that R is an additive group equipped with a multiplication which is associative and unital. The identity element of R is denoted 1_R and is always assumed to be nonzero. We always assume that ring homomorphisms respect the multiplicative identities. The set of ring endomorphisms of R is denoted $\text{End}(R)$ and the center of R is denoted $Z(R)$. By the commutant of a subset S of a ring R , denoted $C_R(S)$, we mean the set of elements of R that commute with each element of S .

Suppose that R_1 is a subring of R i.e. that there is an injective ring homomorphism $R_1 \rightarrow R$. Recall that if R_1 is commutative, then it is called a maximal commutative subring of R if it coincides with its commutant in R . A lot of work has been devoted to investigating the connection between on the one hand maximal commutativity of R_1 in R and on the other hand nonemptiness of intersections of R_1 with nonzero twosided ideals of R (see [2], [4], [5], [8], [9], [10], [18] and [26]). Recently (see [21], [22], [23], [24] and [25]) such a connection was established for the commutant R_1 of the identity component of strongly group graded rings and group crossed products (see Theorem 1 and Theorem 2 below). Let G be a group with identity element e . Recall that a ring R is graded by the group G if there is a set of additive subgroups, R_s , $s \in G$, of R such that $R = \bigoplus_{s \in G} R_s$ and $R_s R_t \subseteq R_{st}$, $s, t \in G$; if $R_s R_t = R_{st}$, $s, t \in G$, then R is called strongly graded. The subring R_e of R is called the identity component of R . The following result appears in [25].

Theorem 1. *If a strongly group graded ring has commutative identity component, then the intersection of a nonzero twosided ideal of the ring with the commutant of the principal component in the ring is nonzero.*

Recall that crossed products are defined by first specifying a crossed system i.e. a quadruple $\{A, G, \sigma, \alpha\}$ where A is a ring, G is a group and $\sigma : G \rightarrow \text{End}(A)$ and $\alpha : G \times G \rightarrow A$ are maps satisfying the following four conditions:

- (1) $\sigma_e = \text{id}_A$
- (2) $\alpha(s, e) = \alpha(e, s) = 1_A$
- (3) $\alpha(s, t)\alpha(st, r) = \sigma_s(\alpha(t, r))\alpha(s, tr)$
- (4) $\sigma_s(\sigma_t(a))\alpha(s, t) = \alpha(s, t)\sigma_{st}(a)$

for all $s, t, r \in G$ and all $a \in A$. The crossed product, denoted $A \rtimes_\alpha^\sigma G$, associated to this quadruple, is the collection of formal sums $\sum_{s \in G} a_s u_s$, where $a_s \in A$, $s \in G$, are chosen so that all but finitely many of them are zero. By abuse of notation we write u_s instead of $1u_s$ for all $s \in G$. The addition on $A \rtimes_\alpha^\sigma G$ is defined pointwise

$$(5) \quad \sum_{s \in G} a_s u_s + \sum_{s \in G} b_s u_s = \sum_{s \in G} (a_s + b_s) u_s$$

and the multiplication on $A \rtimes_\alpha^\sigma G$ is defined by the bilinear extension of the relation

$$(6) \quad (a_s u_s)(b_t u_t) = a_s \sigma_s(b_t) \alpha(s, t) u_{st}$$

for all $s, t \in G$ and all $a_s, b_t \in A$. By (1) and (2) u_e is a multiplicative identity of $A \rtimes_\alpha^\sigma G$ and by (3) the multiplication on $A \rtimes_\alpha^\sigma G$ is associative. There is also an A -bimodule structure on $A \rtimes_\alpha^\sigma G$ defined by the linear extension of the relations $a(bu_s) = (ab)u_s$ and $(au_s)b = (a\sigma_s(b))u_s$ for all $a, b \in A$ and all $s, t \in G$, which, by (4), makes $A \rtimes_\alpha^\sigma G$ an A -algebra. Note that $A \rtimes_\alpha^\sigma G$ is a group graded ring with the grading $(A \rtimes_\alpha^\sigma G)_s = Au_s$, $s \in G$; it is clear that this makes $A \rtimes_\alpha^\sigma G$ a strongly graded ring if and only if each $\alpha(s, t)$, $s, t \in G$, has a left inverse in A . In [24], the following result was shown.

Theorem 2. *If $A \rtimes_\alpha^\sigma G$ is a crossed product with A commutative, all σ_s , $s \in G$, are ring automorphisms and none of the $\alpha(s, s^{-1})$, $s \in G$, are zero divisors in A , then every intersection of a nonzero twosided ideal of $A \rtimes_\alpha^\sigma G$ with the commutant of A in $A \rtimes_\alpha^\sigma G$ is nonzero.*

For more details concerning group graded rings in general and crossed product algebras in particular, see e.g. [1], [7] and [15].

Many natural examples of rings, such as rings of matrices, crossed product algebras defined by separable extensions and category rings, are not in any natural way graded by groups, but instead by categories (see [11], [12],

[13] and [20]). The main purpose of this article is to obtain a simultaneous generalization (see Theorem 3) of Theorem 1 and Theorem 2 as well as extending the result from gradings defined by groups to groupoids. To be more precise, suppose that G is a category. The family of objects of G is denoted $\text{ob}(G)$; we will often identify an object in G with its associated identity morphism. The family of morphisms in G is denoted $\text{mor}(G)$; by abuse of notation, we will often write $s \in G$ when we mean $s \in \text{mor}(G)$. The domain and codomain of a morphism s in G is denoted $d(s)$ and $c(s)$ respectively. We let $G^{(2)}$ denote the collection of composable pairs of morphisms in G i.e. all (s, t) in $\text{mor}(G) \times \text{mor}(G)$ satisfying $d(s) = c(t)$. A category is called a groupoid if all its morphisms are invertible. Recall from [12] that a ring R is called graded by the category G if there is a set of additive subgroups, R_s , $s \in G$, of R such that $R = \bigoplus_{s \in G} R_s$ and for all $s, t \in G$, we have $R_s R_t \subseteq R_{st}$ if $(s, t) \in G^{(2)}$ and $R_s R_t = \{0\}$ otherwise; if $R_s R_t = R_{st}$, $(s, t) \in G^{(2)}$, then R is called strongly graded. By the principal component of R we mean the set $R_0 := \bigoplus_{e \in \text{ob}(G)} R_e$. We say that R has the nonzero ideal property if to each isomorphism $s \in G$ and each nonzero $x \in R_s$, the right R_0 -ideal $xR_{s^{-1}}$ is nonzero. In Section 2, we prove the following result.

Theorem 3. *If a groupoid graded ring has the nonzero ideal property, then the intersection of a nonzero twosided ideal of the ring, with the commutant of the center of the principal component of the ring, is nonzero.*

Theorem 3 generalizes Theorem 1 and Theorem 2. In fact, suppose that R is a ring graded by the group G and that we have chosen $s \in G$ and a nonzero $x \in R_s$. If R is strongly graded, then $0 \neq x = x1_R \in xR_{s^{-1}}R_s$, which implies that the right R_0 -ideal $xR_{s^{-1}}$ is nonzero. Now suppose that R is a group graded crossed product $A \rtimes_\alpha^\sigma G$. Then $x = a_s u_s$ for some nonzero $a_s \in A_e$. Hence $a_s \alpha(s, s^{-1}) = x u_{s^{-1}} \in xR_{s^{-1}}$. Therefore the right R_0 -ideal $xR_{s^{-1}}$ is nonzero as long as $\alpha(s, s^{-1})$ is not a zero divisor in A_e .

In Section 3, we generalize [19, Theorem 3.4], [20, Corollary 6] and [20, Proposition 10] by proving the following result.

Theorem 4. *If $A \rtimes^\sigma G$ is a skew groupoid algebra with all A_e , for $e \in \text{ob}(G)$, commutative rings and $|\text{ob}(G)| < \infty$, then A is maximal commutative in $A \rtimes^\sigma G$ if and only if every intersection of a nonzero twosided ideal of $A \rtimes^\sigma G$ and A is nonzero.*

The secondary purpose of this article is to determine the center of strongly groupoid graded rings R in terms of a groupoid action on R defined by the grading (see Theorem 6 in Section 4). This generalizes a result for group graded rings by the first author together with Silvestrov, Theohari-Apostolidi and Vavatsoulas (see Lemma 3(iii) in [25]) to the groupoid graded situation.

In Section 5, we show that the class of strongly groupoid graded rings which are not crossed products, in the sense defined in [20], is nonempty for any given groupoid with a finite number of objects. In fact, we show, by an explicit construction, the following result.

Theorem 5. *Given a finite groupoid G , which has a nonidentity morphism, there is a ring, strongly graded by G , which is not a crossed product over G .*

2. IDEALS

In this section, we prove Theorem 3 and a corollary thereof. To this end, and for use in the next section, we gather some fairly well known facts from folklore concerning graded rings in a lemma (see also [12] and [15]). We also show that the commutant of the principal component of rings graded by cancellable categories, is again a graded ring (see Proposition (1)); this result will be used in Section 3.

Lemma 1. *Suppose that R is a ring graded by a cancellable category G . We use the notation $1_R = \sum_{s \in G} 1_s$ where $1_s \in R_s$, $s \in G$. (a) $1_R \in R_0$; (b) if we let H denote the set of $s \in G$ with $1_{d(s)} \neq 0 \neq 1_{c(s)}$, then H is a subcategory of G with finitely many objects and $R = \bigoplus_{s \in H} R_s$; (c) if G is a groupoid (or group), then H is a groupoid (or group); (d) if $s \in G$ is an isomorphism, then $R_s R_{s^{-1}} = R_{c(s)}$ if and only if $R_s R_t = R_{st}$ for all $t \in G$ with $d(s) = c(t)$. In particular, if G is a groupoid (or group), then R is strongly graded if and only if $R_s R_{s^{-1}} = R_{c(s)}$, $s \in G$.*

Proof. (a) If $t \in G$, then $1_t = 1_R 1_t = \sum_{s \in G} 1_s 1_t$. Since G is cancellable, this implies that $1_s 1_t = 0$ whenever $s \in G \setminus \text{ob}(G)$. Therefore, if $s \in G \setminus \text{ob}(G)$, then $1_s = 1_s 1_R = \sum_{t \in G} 1_s 1_t = 0$.

(b) Since $d(st) = d(t)$, $c(st) = c(s)$ for all $(s, t) \in G^{(2)}$, we get that H is a subcategory of G . By the fact that $1_R = \sum_{s \in \text{ob}(H)} 1_s$, we get that $\text{ob}(H)$ is finite. Suppose that $s \in G \setminus H$ is chosen such that $1_{c(s)} = 0$. Then $R_s = 1_R R_s = 1_{c(s)} R_s = \{0\}$. The case when $1_{d(s)} = 0$ is treated similarly.

(c) Suppose that G is a groupoid (or group). Since $d(s^{-1}) = c(s)$ and $c(s^{-1}) = d(s)$, $s \in G$, it follows that H is a subgroupoid (or subgroup) of G .

(d) The "if" statement is clear. Now we show the "only if" statement. Take $(s, t) \in G^{(2)}$ and suppose that $R_s R_{s^{-1}} = R_{c(s)}$. Then, by (a), we get that $R_s R_t \subseteq R_{st} = R_{c(s)} R_{st} = R_s R_{s^{-1}} R_{st} \subseteq R_s R_{s^{-1}st} = R_s R_t$. Therefore, $R_s R_t = R_{st}$. The last part follows immediately. \square

Proposition 1. *Suppose that R is a ring graded by a category G and that A is a graded additive subgroup of R . For each $s \in G$, denote $C_R(A)_s := C_R(A) \cap R_s$. If $s, t \in G$, then*

- (a) $C_R(A)_s = \bigcap_{u \in G} C_{R_s}(A_u)$;
- (b) $C_R(A)_s C_R(A)_t \subseteq \begin{cases} C_R(A)_{st}, & \text{if } (s, t) \in G^{(2)}, \\ \{0\}, & \text{otherwise;} \end{cases}$
- (c) $C_R(R_0)$ is a graded subring of R with

$$C_R(R_0)_s = \begin{cases} C_{R_s}(R_{d(s)}), & \text{if } c(s) = d(s), \\ \{r_s \in R_s \mid R_{c(s)} r_s = r_s R_{d(s)} = \{0\}\}, & \text{otherwise;} \end{cases}$$

(d) if $1_R \in R_0$, then $C_R(R_0)$ is a graded subring of R with

$$C_R(R_0)_s = \begin{cases} C_{R_s}(R_{d(s)}), & \text{if } c(s) = d(s), \\ \{0\}, & \text{otherwise.} \end{cases}$$

In particular, if G is cancellable, then the same conclusion holds.

Proof. (a) This is a consequence of the following chain of equalities $C_R(A)_s = C_R(A) \cap R_s = C_{R_s}(A) = C_{R_s}(\bigoplus_{u \in G} A_u) = \bigcap_{u \in G} C_{R_s}(A_u)$.

(b) Suppose that $u \in G$, $a_u \in A_u$, $(s, t) \in G^{(2)}$ and that $r_s \in C_R(A)_s$ and $r'_t \in C_R(A)_t$. Then $(r_s r'_t) a_u = r_s (r'_t a_u) = r_s (a_u r'_t) = (r_s a_u) r'_t = (a_u r_s) r'_t = a_u (r_s r'_t)$. Therefore $r_s r'_t \in C_{R_{st}}(A_u)$ for all $u \in G$. Hence $r_s r'_t \in C_R(A)_{st}$. If, on the other hand, $(s, t) \notin G^{(2)}$, then, by (a), we get that $C_R(A)_s C_R(A)_t \subseteq R_s R_t = \{0\}$.

(c) It is clear that $C_R(R_0) \supseteq \bigoplus_{s \in G} C_R(R_0)_s$. Now we show the reversed inclusion. Take $x \in C_R(R_0)$, $e \in \text{ob}(G)$ and $a_e \in R_e$. Then $\sum_{s \in G} x_s a_e = \sum_{s \in G} a_e x_s$. By comparing terms of the same degree, we can conclude that $x_s a_e = a_e x_s$ for all $s \in G$. Since $e \in \text{ob}(G)$ and $a_e \in A_e$ were arbitrarily chosen this implies that $x_s \in C_R(R_0)_s$ for all $s \in G$. Now we show the second part of (c). Take $e \in \text{ob}(G)$. Suppose that $c(s) = d(s)$. If $d(s) \neq e$, then $C_{R_s}(R_e) = R_s$. Hence $\bigcap_{e \in \text{ob}(G)} C_{R_s}(R_e) = C_{R_s}(R_{d(s)})$. Now suppose that $c(s) \neq d(s)$. If $c(s) \neq e \neq d(s)$, then $C_{R_s}(R_e) = R_s$. Therefore, $\bigcap_{e \in \text{ob}(G)} C_{R_s}(R_e) = C_{R_s}(R_{c(s)}) \cap C_{R_s}(R_{d(s)})$; $C_{R_s}(R_{c(s)})$ equals the set of $r_s \in R_s$ such that $ar_s = r_s a$ for all $a \in R_{c(s)}$. Since $d(s) \neq c(s)$, we get that $r_s a_e = 0$; $C_{R_s}(R_{d(s)})$ is treated similarly.

(d) The claims follow immediately from (c). In fact, suppose that $c(s) \neq d(s)$. Take $r_s \in R_s$ such that $R_{c(s)} r_s = \{0\}$. Then $r_s = 1_R r_s = 1_{c(s)} r_s = 0$. If G is cancellable, then, by Lemma 1(a), the multiplicative identity of R belongs to R_0 . \square

Proof of Theorem 3. We prove the contrapositive statement. Let C denote the commutant of $Z(R_0)$ in R and suppose that I is a twosided ideal of R with the property that $I \cap C = \{0\}$. We wish to show that $I = \{0\}$. Take $x \in I$. If $x \in C$, then by the assumption $x = 0$. Therefore we now assume that $x = \sum_{s \in G} x_s \in I$, $x_s \in R_s$, $s \in G$, and that x is chosen so that $x \notin C$ with the set $S := \{s \in G \mid x_s \neq 0\}$ of least possible cardinality N . Seeking a contradiction, suppose that N is positive. First note that there is $e \in \text{ob}(G)$ with $1_e x \in I \setminus C$. In fact, if $1_e x \in C$ for all $e \in \text{ob}(G)$, then $x = 1_R x = \sum_{e \in \text{ob}(G)} 1_e x \in C$ which is a contradiction. Note that, by Lemma 1(b), the sum $\sum_{e \in \text{ob}(G)} 1_e x$, and hence the sum $\sum_{e \in \text{ob}(G)} 1_e x$, is finite. By minimality of N , we can assume that $c(s) = e$, $s \in S$, for some fixed $e \in \text{ob}(G)$. Take $t \in S$. By the nonzero ideal property there is $y \in R_{t-1}$ with $x_t y \neq 0$. By minimality of N , we can therefore from now on assume that $e \in S$ and $d(s) = c(s) = e$ for all $s \in S$. Take $d = \sum_{f \in \text{ob}(G)} d_f \in Z(R_0)$ where $d_f \in R_f$, $f \in \text{ob}(G)$ and note that $Z(R_0) = \bigoplus_{f \in \text{ob}(G)} Z(R_f)$. Then $I \ni dx - xd = \sum_{s \in S} \sum_{f \in \text{ob}(G)} (d_f x_s - x_s d_f) = \sum_{s \in S} d_e x_s - x_s d_e$. In the R_e

component of this sum we have $d_e x_e - x_e d_e = 0$ since $d_e \in Z(R_e)$. Thus, the summand vanishes for $s = e$, and hence, by minimality of N , we get that $dx - xd = 0$. Since $d \in Z(R_0)$ was arbitrarily chosen, we get that $x \in C$ which is a contradiction. Therefore $N = 0$ and hence $S = \emptyset$ which in turn implies that $x = 0$. Since $x \in I$ was arbitrarily chosen, we finally get that $I = \{0\}$. \square

Corollary 1. *If a groupoid graded ring has the nonzero ideal property and the principal component of the ring is maximal commutative, then the intersection of a nonzero twosided ideal of the ring with the principal component of the ring is nonzero.*

Proof. This follows immediately from Theorem 3. \square

3. SKEW CATEGORY ALGEBRAS

We shall recall the definition of a skew category ring from [20]. By a skew system we mean a triple $\{A, G, \sigma\}$ where G is an arbitrary small category, A is the direct sum of rings A_e , $e \in \text{ob}(G)$, $\sigma_s : A_{d(s)} \rightarrow A_{c(s)}$, for $s \in G$, are ring homomorphisms, satisfying the following two conditions:

$$(7) \quad \sigma_e = \text{id}_{A_e}$$

$$(8) \quad \sigma_s(\sigma_t(a)) = \sigma_{st}(a)$$

for all $e \in \text{ob}(G)$, all $(s, t) \in G^{(2)}$ and all $a \in A_{d(t)}$. Let $A \rtimes^\sigma G$ denote the collection of formal sums $\sum_{s \in G} a_s u_s$, where $a_s \in A_{c(s)}$, $s \in G$, are chosen so that all but finitely many of them are zero. Define addition on $A \rtimes^\sigma G$ by (5) and define multiplication on $A \rtimes^\sigma G$ as the bilinear extension of the relation

$$(9) \quad (a_s u_s)(b_t u_t) = a_s \sigma_s(b_t) u_{st}$$

if $(s, t) \in G^{(2)}$ and $(a_s u_s)(b_t u_t) = 0$ otherwise where $a_s \in A_{c(s)}$ and $b_t \in A_{c(t)}$. One can show that $A \rtimes^\sigma G$ has a multiplicative identity if and only if $\text{ob}(G)$ is finite; in that case the multiplicative identity is $\sum_{e \in \text{ob}(G)} u_e$. It is easy to verify that the multiplication on $A \rtimes^\sigma G$ is associative. Define a left A -module structure on $A \rtimes^\sigma G$ by the bilinear extension of the rule $a_e(b_s u_s) = (a_e b_s) u_s$ if $e = c(s)$ and $a_e(b_s u_s) = 0$ otherwise for all $a_e \in A_e$, $b_s \in A_{c(s)}$, $e \in \text{ob}(G)$, $s \in G$. Analogously, define a right A -module structure on $A \rtimes^\sigma G$ by the bilinear extension of the rule $(b_s u_s)c_f = (b_s \sigma_s(c_f)) u_s$ if $f = d(s)$ and $(b_s u_s)c_f = 0$ otherwise for all $b_s \in A_{c(s)}$, $c_f \in A_f$, $f \in \text{ob}(G)$, $s \in G$. By (8) this A -bimodule structure makes $A \rtimes_\alpha^\sigma G$ an A -algebra. We will often identify A with $\bigoplus_{e \in \text{ob}(G)} A_e u_e$; this ring will be referred to as the coefficient ring or principal component of $A \rtimes_\alpha^\sigma G$. It is clear that $A \rtimes_\alpha^\sigma G$ is a category graded ring in the sense of [12] and it is strongly graded. We call $A \rtimes^\sigma G$ the *skew category algebra* associated to the skew system $\{A, G, \sigma\}$.

Proposition 2. *If $A \rtimes^\sigma G$ is a skew category algebra with all A_e , for $e \in \text{ob}(G)$, commutative rings and $|\text{ob}(G)| < \infty$, such that every intersection of a nonzero twosided ideal of $A \rtimes^\sigma G$ and A is nonzero, then A is maximal commutative in $A \rtimes^\sigma G$.*

Proof. We show the contrapositive statement. Suppose that A is not maximal commutative in $A \rtimes^\sigma G$. Then, by Proposition 1(d), there exists some $s \in G \setminus G_0$, with $d(s) = c(s)$, and some nonzero $r_s \in A_{c(s)}$, such that $r_s u_s$ commutes with all of A . Let I be the (nonzero) ideal in $A \rtimes^\sigma G$ generated by the element $r_s u_{c(s)} - r_s u_s$. Note that all elements of I are sums of elements of the form

$$(10) \quad a_g u_g (r_s - r_s u_s) a_h u_h$$

for $g, h \in G$, $a_g \in A_{c(g)}$ and $a_h \in A_{c(h)}$. Suppose that $(g, h) \in G^{(2)}$, for otherwise the above element is automatically zero. We may now simplify:

$$\begin{aligned} a_g u_g (r_s - r_s u_s) a_h u_h &= a_g \sigma_g(r_s a_h) u_{gh} - a_g \sigma_g(a_h r_s) u_{gsh} \\ &= \underbrace{a_g \sigma_g(r_s a_h)}_{=:b} u_{gh} - \underbrace{a_g \sigma_g(r_s a_h)}_{=:b} u_{gsh} \end{aligned}$$

Consider the additive map

$$\varphi : A \rtimes^\sigma G \rightarrow A, \quad \sum_{s \in G} a_s u_s \mapsto \sum_{s \in G} a_s.$$

It is clear that the restriction of φ to A is injective. And since each element of I is a sum of elements of the form (10) it follows that φ is identically zero on all of I . This shows that $I \cap A = \{0\}$ and hence the desired conclusion follows. \square

Proof of Theorem 4. The "if" statement follows from Theorem 3. The "only if" statement follows from Proposition 2 if we let G be a groupoid. \square

4. THE CENTER

In this section, we determine the center of strongly groupoid graded rings (see Theorem 6) in terms of an action on the ring induced by the grading (see Definition 2). This is established through three propositions formulated in a slightly more general setting.

Proposition 3. *Suppose that R is a ring graded by a category G and that $s \in G$ is an isomorphism. For each positive integer i take $a_s^{(i)}, c_s^{(i)} \in R_s$ and $b_{s^{-1}}^{(i)}, d_{s^{-1}}^{(i)} \in R_{s^{-1}}$ with the property that $a_s^{(i)} = b_{s^{-1}}^{(i)} = c_s^{(i)} = d_{s^{-1}}^{(i)} = 0$ for all but finitely many i . If $x, y \in C_R(R_{s^{-1}} R_s)$ and*

$$A = \sum_{i=1}^{\infty} a_s^{(i)} x y b_{s^{-1}}^{(i)} \sum_{j=1}^{\infty} c_s^{(j)} d_{s^{-1}}^{(j)}$$

$$B = \sum_{i=1}^{\infty} a_s^{(i)} x b_{s-1}^{(i)} \sum_{j=1}^{\infty} c_s^{(j)} y d_{s-1}^{(j)}$$

$$C = \sum_{i=1}^{\infty} a_s^{(i)} b_{s-1}^{(i)} \sum_{j=1}^{\infty} c_s^{(j)} x y d_{s-1}^{(j)}$$

then $A = B = C$. In particular, if G is cancellable and

$$\sum_{i=1}^{\infty} a_s^{(i)} b_{s-1}^{(i)} = \sum_{j=1}^{\infty} c_s^{(j)} d_{s-1}^{(j)} = 1_{c(s)}$$

then the following equalities hold

$$(11) \quad \sum_{i=1}^{\infty} a_s^{(i)} x b_{s-1}^{(i)} = \sum_{j=1}^{\infty} c_s^{(j)} x d_{s-1}^{(j)}$$

$$(12) \quad \sum_{i=1}^{\infty} a_s^{(i)} x y b_{s-1}^{(i)} = \sum_{i=1}^{\infty} a_s^{(i)} x b_{s-1}^{(i)} \sum_{i=1}^{\infty} a_s^{(i)} y b_{s-1}^{(i)}$$

Proof. Suppose that $x, y \in C_R(R_{s-1}R_s)$. The equality $A = B$ (or $B = C$) follows from the fact that y (or x) commutes with $b_{s-1}^{(i)} c_s^{(j)}$ for all positive integers i and j . The equality (11) follows from Lemma 1(a) and the equality $A = C$ with $y = 1_{d(s)}$. The equality (12) follows from Lemma 1(a), equality (11) and the equality $A = B$. \square

Proposition 4. Suppose that R is a ring graded by a category G and that $s, t \in G$ are isomorphisms with $d(s) = c(t)$. For each positive integer i take $a_s^{(i)} \in R_s$, $b_{s-1}^{(i)} \in R_{s-1}$, $c_t^{(i)} \in R_t$, $d_{t-1}^{(i)} \in R_{t-1}$, $p_{st}^{(i)} \in R_{st}$ and $q_{(st)^{-1}}^{(i)} \in R_{(st)^{-1}}$ with the property that $a_s^{(i)} = b_{s-1}^{(i)} = c_t^{(i)} = d_{t-1}^{(i)} = p_{st}^{(i)} = q_{(st)^{-1}}^{(i)} = 0$ for all but finitely many i . If $x \in C_R(R_{(st)^{-1}}R_sR_t)$ and

$$D = \sum_{k=1}^{\infty} p_{st}^{(k)} x q_{(st)^{-1}}^{(k)} \sum_{i=1}^{\infty} a_s^{(i)} \sum_{j=1}^{\infty} c_t^{(j)} d_{t-1}^{(j)} b_{s-1}^{(i)}$$

$$E = \sum_{k=1}^{\infty} p_{st}^{(k)} q_{(st)^{-1}}^{(k)} \sum_{i=1}^{\infty} a_s^{(i)} \sum_{j=1}^{\infty} c_t^{(j)} x d_{t-1}^{(j)} b_{s-1}^{(i)}$$

then $D = E$. In particular, if G is cancellable and the following equalities hold

$$\sum_{k=1}^{\infty} p_{st}^{(k)} q_{(st)^{-1}}^{(k)} = \sum_{i=1}^{\infty} a_s^{(i)} b_{s-1}^{(i)} = 1_{c(s)} \quad \sum_{j=1}^{\infty} c_t^{(j)} d_{t-1}^{(j)} = 1_{c(t)}$$

then

$$(13) \quad \sum_{k=1}^{\infty} p_{st}^{(k)} x q_{(st)^{-1}}^{(k)} = \sum_{i=1}^{\infty} a_s^{(i)} \sum_{j=1}^{\infty} c_t^{(j)} x d_{t-1}^{(j)} b_{s-1}^{(i)}$$

Proof. Suppose that $x \in C_R(R_{(st)^{-1}}R_sR_t)$. The equality $D = E$ follows from the fact that x commutes with $q_{(st)^{-1}}^{(k)}a_s^{(i)}c_t^{(j)}$ for all positive integers i, j and k . The equality (13) follows from Lemma 1(a) and the equality $D = E$. \square

Proposition 5. *Suppose that R is a ring graded by a category G and that $s \in G$ is an isomorphism. For each positive integer i take $a_s^{(i)}, c_s^{(i)} \in R_s$ and $b_{s^{-1}}^{(i)}, d_{s^{-1}}^{(i)} \in R_{s^{-1}}$ with the property that $a_s^{(i)} = b_{s^{-1}}^{(i)} = c_s^{(i)} = d_{s^{-1}}^{(i)} = 0$ for all but finitely many i . If $x \in C_R(R_{s^{-1}}R_{c(s)}R_s)$ and $y \in R_{c(s)}$, then*

$$(14) \quad \sum_{i=1}^{\infty} a_s^{(i)} x b_{s^{-1}}^{(i)} y \sum_{j=1}^{\infty} c_s^{(j)} d_{s^{-1}}^{(j)} = \sum_{i=1}^{\infty} a_s^{(i)} b_{s^{-1}}^{(i)} y \sum_{j=1}^{\infty} c_s^{(j)} x d_{s^{-1}}^{(j)}$$

In particular, if G is cancellable and

$$\sum_{i=1}^{\infty} a_s^{(i)} b_{s^{-1}}^{(i)} = \sum_{j=1}^{\infty} c_s^{(j)} d_{s^{-1}}^{(j)} = 1_{c(s)}$$

then

$$(15) \quad \sum_{i=1}^{\infty} a_s^{(i)} x b_{s^{-1}}^{(i)} \in C_R(R_{c(s)})$$

If also $x \in Z(R_{d(s)})$, then

$$(16) \quad \sum_{i=1}^{\infty} a_s^{(i)} x b_{s^{-1}}^{(i)} \in Z(R_{c(s)})$$

Proof. Suppose that $x \in C_R(R_{s^{-1}}R_{c(s)}R_s)$ and $y \in R_{c(s)}$. The equality (14) follows from the fact that x commutes with $b_{s^{-1}}^{(i)} y c_s^{(j)}$ for all positive integers i and j . The claim (15) follows from (11) and (14). The claim (16) follows from (15) and the fact that $Z(R_e) = R_e \cap C_R(R_e)$ for any $e \in \text{ob}(G)$. \square

Definition 1. Suppose that R is a ring strongly graded by a groupoid G . By abuse of notation, we let $C(R)$ (or $Z(R)$) denote the groupoid with $C_R(R_e)$ (or $Z(R_e)$), $e \in \text{ob}(G)$, as objects, and the ring isomorphisms $C_R(R_{d(s)}) \rightarrow C_R(R_{c(s)})$ (or $Z(R_{d(s)}) \rightarrow Z(R_{c(s)})$), $s \in G$, as morphisms.

Definition 2. Suppose that R is a ring strongly graded by a groupoid G . For each $s \in G$ and each positive integer i , take $a_s^{(i)} \in R_s$ and $b_{s^{-1}}^{(i)} \in R_{s^{-1}}$ with the property that $a_s^{(i)} = b_{s^{-1}}^{(i)} = 0$ for all but finitely many i and $\sum_{i=1}^{\infty} a_s^{(i)} b_{s^{-1}}^{(i)} = 1_{c(s)}$. Define a function $\sigma_s : R \rightarrow R$ by $\sigma_s(x) = \sum_{i=1}^{\infty} a_s^{(i)} x b_{s^{-1}}^{(i)}$, $x \in R$. By abuse of notation, we let every restriction of σ_s to subsets of R also be denoted by σ_s .

Proposition 6. *Suppose that R is a ring strongly graded by a groupoid G . Then the association of each $e \in \text{ob}(G)$ and each $s \in G$ to the ring $C_R(R_e)$ (or the ring $Z(R_e)$) and the function $\sigma_s : C_R(R_{d(s)}) \rightarrow C_R(R_{c(s)})$ (or $\sigma_s : Z(R_{d(s)}) \rightarrow Z(R_{c(s)})$) respectively, defines a functor of groupoids*

$\sigma : G \rightarrow C(R)$ (or $\sigma : G \rightarrow Z(R)$). Moreover, σ is uniquely defined on morphisms by the relations $\sigma_s(x)r_s = r_sx$ and $\sigma_s(x)1_{c(s)} = \sigma_s(x)$, $s \in G$, $x \in C_R(R_{d(s)})$ (or $x \in Z(R_{d(s)})$), $r_s \in R_s$.

Proof. We show the claim about $C(R)$. Since the claim about $Z(R)$ can be shown in a completely analogous way we leave the details of this to the reader. Take $s \in G$. By (15), σ_s is well defined. It is clear that σ_s is additive and that $\sigma_s(1_{R_{d(s)}}) = 1_{R_{c(s)}}$. By (12), σ_s is multiplicative. Take $t \in G$ with $d(s) = c(t)$. By (13), $\sigma_{st} = \sigma_s\sigma_t$. By (11), the definition of σ_s does not depend on the particular choice of $a_s^{(i)} \in R_s$ and $b_{s^{-1}}^{(i)} \in R_{s^{-1}}$ subject to the condition $\sum_{i=1}^{\infty} a_s^{(i)} b_{s^{-1}}^{(i)} = 1_{c(s)}$. Therefore, for each $e \in \text{ob}(G)$, we can make the choice $a_e^{(1)} = b_e^{(1)} = 1_e$ and $a_e^{(i)} = b_e^{(i)} = 0$ for all integers $i \geq 2$; it is easy to see that this implies that $\sigma_e = \text{id}_{C_R(R_e)}$. For the second part of the proof suppose that $s \in G$, $x \in C_R(R_{d(s)})$ and $y \in R$ satisfy $yr_s = r_sx$ for all $r_s \in R_s$. Then $\sigma_s(x) = \sum_{i=1}^{\infty} a_s^{(i)} x b_{s^{-1}}^{(i)} = \sum_{i=1}^{\infty} y a_s^{(i)} b_{s^{-1}}^{(i)} = y 1_{c(s)} = y$. Finally, suppose that $s \in G$, $x \in C_R(R_{d(s)})$ and $r_s \in R_s$. Then $\sigma_s(x)r_s = \sum_{i=1}^{\infty} a_s^{(i)} x b_{s^{-1}}^{(i)} r_s = \sum_{i=1}^{\infty} a_s^{(i)} b_{s^{-1}}^{(i)} r_s x = 1_{c(s)} r_s x = r_s x$. It is clear that $\sigma_s(x)1_{c(s)} = \sigma_s(x)$. \square

Theorem 6. *The center of a strongly groupoid graded ring R equals the collection of $\sum_{e \in \text{ob}(G)} x_e$, $x_e \in C_R(R_e)$, $e \in \text{ob}(G)$, satisfying $\sigma_s(x_{d(s)}) = x_{c(s)}$, $s \in G$. In particular, if G is the disjoint union of groups G_e , $e \in \text{ob}(G)$, then the center of R equals the direct sum of the rings $C_R(R_e)^{G_e}$, $e \in \text{ob}(G)$.*

Proof. Suppose that $y = \sum_{s \in G} y_s$ belongs to the center of R where $y_s \in R_s$, $s \in G$, and $y_s = 0$ for all but finitely many $s \in G$. Since $1_e y = y 1_e$, $e \in \text{ob}(G)$, we get that $y_s = 0$ whenever $c(s) \neq d(s)$. Therefore, $y = \sum_{e \in \text{ob}(G)} x_e$ where $x_e = \sum_{s \in G_e} y_s$, $e \in \text{ob}(G)$. Since $y \in C_R(R_e)$, $e \in \text{ob}(G)$, we get that $x_e \in C_R(R_e)$, $e \in \text{ob}(G)$. Take $s \in G$. The relation $r_s y = y r_s$, $r_s \in R_s$, and the last part of Proposition 6 imply that $\sigma_s(x_{d(s)}) = x_{c(s)}$. On the other hand, it is clear, by the last part of Proposition 6, that all sums $\sum_{e \in \text{ob}(G)} x_e$, $x_e \in C_R(R_e)$, $e \in \text{ob}(G)$, satisfying $\sigma_s(x_{d(s)}) = x_{c(s)}$, $s \in G$, belong to the center of R . The last part of the claim is obvious. \square

5. EXAMPLES

In this section, we show Theorem 5. Our method will be to generalize, to category graded rings (see Proposition 7), the construction given in [3] for the group graded situation. To do that, we first need to introduce some more notations. Let K be a field and G a category. Suppose that n is a positive integer and choose $s_i \in G$, for $1 \leq i \leq n$. For $1 \leq i, j \leq n$, let $e_{ij} \in M_n(K)$ be the matrix with 1 in the ij th position and 0 elsewhere. For $s \in G$, we let R_s be the K -vector subspace of $M_n(K)$ spanned by the set of e_{ij} , for $1 \leq i, j \leq n$, such that $(s_i, s) \in G^{(2)}$ and $s_i s = s_j$.

Proposition 7. *If $s, t \in G$, then, with the above notations, we get that*

- (a) $R_s R_t \subseteq R_{st}$, if $(s, t) \in G^{(2)}$, and $R_s R_t = \{0\}$, otherwise.
- (b) If G is cancellable, then the sum $R := \sum_{s \in G} R_s$ is direct. Therefore, in that case, R is a ring graded by G with components R_s , for $s \in G$.

Proof. (a) Suppose that $(s, t) \in G^{(2)}$. Take $e_{ij} \in R_s$ and $e_{lk} \in R_t$. If $j \neq l$, then $e_{ij}e_{lk} = 0 \in R_{st}$. Now let $j = l$. Then, since $s_i s = s_j$ and $s_j t = s_k$, we get that $s_i s t = s_j t = s_k$. Hence, $e_{ij}e_{jk} = e_{ik} \in R_{st}$.

(b) Let X_s denote the collection of pairs (i, j) , where $1 \leq i, j \leq n$, such that $(s_i, s) \in G^{(2)}$ and $s_i s = s_j$. Suppose that $s \neq t$. Seeking a contradiction suppose that $X_s \cap X_t \neq \emptyset$. Then there would be integers k and l , with $1 \leq k, l \leq n$, such that $s_k s = s_l = s_k t$. By the cancellability of G this would imply that $s = t$. Therefore, the sets X_s , for $s \in G$, are pairwise disjoint. The claim now follows from the fact that $R_s = \sum_{(i,j) \in X_s} K e_{ij}$ for all $s \in G$. \square

Proof of Theorem 5. Let H be a finite connected groupoid with at least one nonidentity morphism. We begin by showing that one may always find a subring of a matrix ring which is strongly graded by H , but which is not a crossed product in the sense of [20]. If H only has one object, then it is a group in which case it has already been treated in [3]. Therefore, from now on, we assume that we can choose two different objects e and f from H . We denote the morphisms of H by t_1, t_2, \dots, t_n . For technical reasons, we suppose that $t_n = e$ and that $d(t_1) = f$ and $c(t_1) = e$. Let us now choose $n+1$ morphisms s_1, s_2, \dots, s_{n+1} from H in the following way; $s_i = t_i$, when $1 \leq i \leq n$, and $s_{n+1} = t_n$.

First we define R according to the beginning of Section 5 and show that it is strongly graded by H . Take $(s, t) \in H^{(2)}$ and $e_{ki} \in R_{st}$. This means that $s_i s t = s_k$. Now pick an integer j with $1 \leq j \leq n$ and $s_i t = s_j$; this is possible since $\{s_i \mid 1 \leq i \leq n\} = H$. Then $e_{ji} \in R_s$ and, since $s_j s = s_i t s = s_k$, we get that $e_{kj} \in R_s$. Therefore, $e_{ki} = e_{kj}e_{ji} \in R_s R_t$. Hence, R is strongly graded.

Now we shall show that R is not a crossed product over H in the sense defined in [20]. For $g, h \in \text{ob}(H)$, let $H_{g,h}$ denote the set of $s \in H$ with $c(s) = g$ and $d(s) = h$. Since H is connected, all the sets $H_{g,h}$ have the same cardinality; denote this cardinality by m . The component R_e is the K -vector space spanned by the collection of e_{ij} with $s_i e = s_j$, that is, such that $s_i = s_j$ and $d(s_j) = e$. Therefore, the K -dimension of R_e equals $m+3$. Furthermore, the component R_{t_1} is the K -vector space spanned by the collection of e_{ij} with $s_i t_1 = s_j$. Since $d(t_1) = f \neq e$, this implies that the K -dimension of R_{t_1} equals $m+1$. Seeking a contradiction, suppose that R_{t_1} is free on one generator u as a left R_e -module. Then the map $\theta : R_e \rightarrow R_{t_1}$, defined by $\theta(x) = xu$, for $x \in R_e$, is, in particular, an isomorphism of K -vector spaces. Since $\dim_K(R_e) = m+3 > m+1 = \dim_K(R_{t_1})$, this is impossible.

We shall now show that our groupoid G is the disjoint union of connected groupoids. Define an equivalence relation \sim on $\text{ob}(G)$ by saying that $e \sim f$, for $e, f \in \text{ob}(G)$, if there is a morphism in G from e to f . Choose a set E of representatives for the different equivalence classes defined by \sim . For

each $e \in E$, let $[e]$ denote the equivalence class to which e belongs. Let $G_{[e]}$ denote the subgroupoid of G with $[e]$ as set of objects and morphisms $s \in G$ with the property that $c(s), d(s) \in [e]$. Then each $G_{[e]}$, for $e \in E$, is a connected groupoid and $G = \bigsqcup_{e \in E} G_{[e]}$. For each $e \in E$, we now wish to define a strongly $G_{[e]}$ -graded ring $R_{[e]}$. We consider three cases. If $G_{[e]} = \{e\}$, then let $R_{[e]} = K$. If $[e] = \{e\}$ but the group $G_{[e]}$ contains a nonidentity morphism, then let $R_{[e]}$ be a strongly $G_{[e]}$ -graded ring which is not a crossed product as defined in [3]. If $[e]$ has more than one element, let $R_{[e]}$ denote the strongly $G_{[e]}$ -graded ring according to the construction in the first part of the proof. We may define a new ring S to be the direct sum $\bigoplus_{e \in E} R_{[e]}$ and one concludes that S is strongly graded by G but not a crossed product in the sense of [20]. \square

Example 1. To exemplify Theorem 5, we now give explicitly the construction in the simplest possible case when G is not a group. Namely, let G be the unique thin¹ connected groupoid with two objects. More concretely this means that the morphisms of G are e, f, s and t ; multiplication is defined by the relations

$$e^2 = e, \quad f^2 = f, \quad es = s, \quad te = t, \quad sf = s, \quad ft = t.$$

Put

$$s_1 = e, \quad s_2 = s, \quad s_3 = t, \quad s_4 = s_5 = f.$$

and define the G -graded ring R as above. A straightforward calculation shows that

$$\begin{aligned} R_e &= Ke_{11} + Ke_{33} \\ R_f &= Ke_{22} + Ke_{44} + Ke_{45} + Ke_{54} + Ke_{55} \\ R_s &= Ke_{12} + Ke_{34} + Ke_{35} \\ R_t &= Ke_{21} + Ke_{43} + Ke_{53} \end{aligned}$$

Another straightforward calculation shows that

$$\begin{aligned} R_e R_e &= R_e, & R_f R_f &= R_f, & R_e R_s &= R_s \\ R_t R_e &= R_t, & R_s R_f &= R_s, & R_f R_t &= R_t. \end{aligned}$$

Therefore R is strongly graded by G . However, R is not a crossed product in the sense defined in [20]. In fact, since $\dim_K(R_f) = 5 > 3 = \dim_K(R_t)$, the left R_f -module R_t can not be free on one generator. By a similar argument, the left R_e -module R_s is not free on one generator.

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¹In the sense that there is at most one morphism from one object to another.

REFERENCES

- [1] S. Caenepeel, F. Van Oystaeyen, Brauer groups and the cohomology of graded rings. Monographs and Textbooks in Pure and Applied Mathematics, 121. Marcel Dekker, Inc., New York (1988).
- [2] M. Cohen, S. Montgomery, Group-Graded Rings, Smash Products and Group Actions, *Trans. Amer. Math. Soc.* **282**, no. 1, 237–258 (1984).
- [3] S. Dăscălescu, B. Ion, C. Năstăsescu and J. Rios Montes, Group Gradings on Full Matrix Rings, *J. Algebra* **220**, 709–728 (1999).
- [4] J. W. Fisher, S. Montgomery, Semiprime Skew Group Rings, *J. Algebra* **52**, no. 1, 241–247 (1978).
- [5] E. Formanek, A. I. Lichtman, Ideals in Group Rings of Free Products, *Israel J. Math.* **31**, no. 1, 101–104 (1978).
- [6] P. J. Higgins, *Notes on Categories and Groupoids*, Van Nostrand (1971).
- [7] G. Karpilovsky, The Algebraic Structure of Crossed Products, x+348 pp. North-Holland Mathematics Studies, 142. Notas de Matemática, 118. North-Holland, Amsterdam (1987).
- [8] M. Lorenz, D. S. Passman, Centers and Prime Ideals in Group Algebras of Polycyclic-by-Finite Groups, *J. Algebra* **57**, no. 2, 355–386 (1979).
- [9] M. Lorenz, D. S. Passman, Prime Ideals in Crossed Products of Finite Groups, *Israel J. Math.* **33**, no. 2, 89–132 (1979).
- [10] M. Lorenz, D. S. Passman, Addendum - Prime Ideals in Crossed Products of Finite Groups, *Israel J. Math.* **35**, no. 4, 311–322 (1980).
- [11] P. Lundström, Crossed Product Algebras Defined by Separable Extensions, *J. Algebra* **283** (2005), 723–737.
- [12] P. Lundström, Separable Groupoid Rings, *Communications in Algebra* **34** (2006), 3029–3041.
- [13] P. Lundström, The Picard Groupoid and Strongly Groupoid Graded Modules, *Colloquium Mathematicum* **106** (2006), 1–13.
- [14] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics, 5. Springer-Verlag, New York (1998).
- [15] C. Năstăsescu and F. Van Oystaeyen, *Graded Ring Theory*, North-Holland Publishing Co., Amsterdam-New York (1982).
- [16] E. Nauwelaerts, F. Van Oystaeyen, Introducing Crystalline Graded Algebras, *Algebr. Represent. Theory* **11**, no. 2, 133–148 (2008).
- [17] T. Neijens, F. Van Oystaeyen, W. W. Yu, Centers of Certain Crystalline Graded Rings. Preprint in preparation (2007).
- [18] S. Montgomery, D. S. Passman, Crossed Products over Prime Rings, *Israel J. Math.* **31**, no. 3-4, 224–256 (1978).
- [19] J. Öinert, Simple group graded rings and maximal commutativity, arXiv:0904.4661v1 [math.RA]
- [20] J. Öinert, P. Lundström, Commutativity and Ideals in Category Crossed Products, Preprints in Mathematical Sciences 2008:22, LUTFMA-5106-2008, ISSN 1403-9338
- [21] J. Öinert, S. D. Silvestrov, Commutativity and Ideals in Algebraic Crossed Products, *J. Gen. Lie T. Appl.* **2**, no. 4, 287–302 (2008).
- [22] J. Öinert, S. D. Silvestrov, On a Correspondence Between Ideals and Commutativity in Algebraic Crossed Products, *J. Gen. Lie T. Appl.* **2**, no. 3, 216–220 (2008).
- [23] J. Öinert, S. D. Silvestrov, Crossed Product-Like and Pre-Crystalline Graded Rings, 16 pp. in *Generalized Lie Theory in Mathematics, Physics and Beyond*. Conference proceedings of Algebra, Geometry and Mathematical Physics, Baltic-Nordic Workshop (Lund, October 12-14 2006). Springer (2008).
- [24] J. Öinert, S. Silvestrov, Commutativity and Ideals in Pre-Crystalline Graded Rings, Preprints in Mathematical Sciences 2008:17, LUTFMA-5101-2008, ISSN 1403-9338

- [25] J. Öinert, S. Silvestrov, T. Theohari-Apostolidi and H. Vavatsoulas, Commutativity and Ideals in Strongly Graded Rings, Preprints in Mathematical Sciences 2008:13, LUTFMA-5100-2008, ISSN 1403-9338
- [26] D. S. Passman, *The Algebraic Structure of Group Rings*, xiv+720 pp. Pure and Applied Mathematics. Wiley-Interscience (John Wiley & Sons), New York-London-Sydney (1977).
- [27] L. Rowen, Some Results on the Center of a ring with Polynomial Identity, *Bull. Amer. Math. Soc.* **79** (1973), 219–223.
- [28] C. Svensson, S. Silvestrov, M. de Jeu, Dynamical Systems and Commutants in Crossed Products, *Internat. J. Math.* **18**, no. 4, 455–471 (2007).
- [29] C. Svensson, S. Silvestrov, M. de Jeu, Connections between dynamical systems and crossed products of Banach algebras by \mathbb{Z} , to appear in the proceedings of "Operator Theory, Analysis and Mathematical Physics", OTAMP-2006, Lund, Sweden, June 15–22, 2006. (Preprints in Mathematical Sciences 2007:5, LUTFMA-5081-2007; Leiden Mathematical Institute report 2007-02; arXiv:math/0702118).
- [30] C. Svensson, S. Silvestrov, M. de Jeu, Dynamical systems associated with crossed products, to appear in the proceedings of "Operator Methods in Fractal Analysis, Wavelets and Dynamical Systems", BIRS, Banff, Canada, December 2 - December 7, 2006. (Preprints in Mathematical Sciences 2007:22, LUTFMA-5088-2007; Leiden Mathematical Institute report 2007-30; arXiv:0707.1818).
- [31] C. Svensson, J. Tomiyama, On the commutant of $C(X)$ in C^* -crossed products by \mathbb{Z} and their representations. arXiv:0807.2940
- [32] F. Van Oystaeyen, On Clifford Systems and Generalized Crossed Products, *J. Algebra* **87**, 396–415 (1984).

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